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κ symmetric OSp(2|2) WZNW model

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Abstract: We construct a κ symmetric WZNW model for the OSp(2|2) supergroup, whose bosonic part is $AdS_3 \times S^1$ space. The field equation gives the chiral current conservation and the right/left factorization is shown after the κ symmetry is fixed. The right-moving modes contain both bosons and fermions while the left-moving modes contain only bosons.

Keywords: Superstrings and Heterotic Strings, AdS-CFT Correspondence.

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Contents

1. Introduction

Superstrings in anti-de Sitter (AdS) spaces have important feature that those have conformal field theory duals. Recently it has been discussed that the heterotic string theory in AdS_3 space has the dual heterotic nonlinear superconformal algebras [1]. There are many studies on the AdS/CFT correspondence for the type II superstring theories whose worldsheet actions are known such as the σ models given in [2]. The pp-wave limit of the worldsheet type II superstring action allows the lightcone quantization [3]. Further generalization [4] leads to the integrable property of the system which became one of the guiding principles to explore the AdS/CFT correspondence. On the other hand few studies on the ones for a heterotic string have been done where an worldsheet action of a heterotic string in AdS space is not known so far.

The heterotic string is a combination of a chiral bosonic string and a chiral superstring. A chiral superstring in flat space is well described by the Neveu-Schwarz-Ramond formulation, but it is difficult to describe spacetime supersymmetry in curved space because of the lack of the spacetime fermions. A superstring in curved space is well described by the Green-Schwarz formulation, but it is difficult to separate chiral right/left-moving modes for the worldsheet superconformal theory. There are several formulations of a chiral superstring in AdS space where chiral spacetime fermions make both spacetime supersymmetry and right/left separation manifest from the beginning such as the supergroup covariant σ models [5–8]. In these formulations the existence of the kinetic term for those fermions avoids the κ symmetry, and the worldsheet conformal field theory technique is available for the quantum computation. The κ symmetry is an inevitable ingredient of the Green-Schwarz formulation in which the kinetic term is made of only bosonic current bilinears, and it is necessary to remove unphysical fermionic degrees of freedom. There is an interesting observation [9] that the κ symmetric AdS strings are integrable and dual field theories are at conformal fixed points. In this paper we also require the κ symmetry and we construct a worldsheet action for a "heterotic" string in AdS space as a WZNW model.

The bosonic string in AdS_3 space was analyzed by the SL(2) WZNW model [10] where chiral right/left separation makes the quantum analysis possible. In this paper we extend this AdS_3 bosonic string to an AdS_3 "heterotic" string. The chiral right/left separation of an abelian σ model is resulted from the current conservation;

$$\epsilon^{\mu\nu}\partial_\mu J_\nu = 0 \ , \quad \partial_\mu J^\mu = 0 \quad \to \quad \partial_+ J_- = 0 = \partial_- J_+ \quad . \tag{1.1}$$

For a non-abelian target space the flatness condition contains the structure constant dependent term. In order to obtain the chiral current conservation the Wess-Zumino (WZ) term is added in such a way that it gives an extra contribution to the current conservation [11];

$$\epsilon^{\mu\nu}(\partial_\mu J_\nu + J_\mu J_\nu) = 0 \ , \quad \partial_\mu (J^\mu + \epsilon^{\mu\nu} J_\nu) = 0 \quad \to \quad \partial_+ J_- = 0 \quad . \tag{1.2}$$

Anti-chiral current conservation may be constructed as $\partial_- \tilde{J}_+ = 0$ where $\tilde{J}_+$ must be a different function from J_+.

For a Green-Schwarz superstring another type of WZ term is required for the κ symmetry [12]. But this WZ term does not change the current conservation equation. For example the superstring in the $AdS_5 \times S^5$ space [2, 13] there exist the non-abelian currents satisfying the flatness condition and the current conservation

$$\epsilon^{\mu\nu}(\partial_\mu J_\nu + J_\mu J_\nu) = 0 \ , \quad \partial_\mu J^\mu = 0 \quad \to \quad \partial_+ J_- = -\partial_- J_+ = -\frac{1}{2} J_{[+} J_{-]} \neq 0 \quad , \tag{1.3}$$

which is the criteria of the integrable system [14]. The equations in (1.3), which is neither (1.1) nor (1.2), do not give a chiral current conservation. The non-abelian bosonic WZ term should be also necessary for the type of equations in (1.2). It is denoted that the currents $J_\pm$ in (1.3) are the right-invariant (RI) currents rather than the left-invariant (LI) currents when the action is written in terms of the LI currents [15]. The supercovariant derivatives, which are combination of the LI currents, are separated into two chiral sectors on the constrained surface satisfying the same Poisson bracket as the one for the right/left sectors in the flat case [16, 17]. The problem how to reflect the chiral separation of the LI currents into the right/left separation of the RI currents will reduce to the problem of the choice of a coordinate system and the gauge fixing. In this paper we construct the correct WZ term which guarantees the κ symmetry and gives chiral currents conservation as (1.2) for OSp(2|2) supergroup as a simplest nontrivial example.

The organization of this paper is the following: in section 2 we review the orthosymplectic supergroup and especially OSp(2|2) in detail which is used throughout this paper. The group structure and parametrization related to the AdS_3 metric are presented. In

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section 3 we propose a κ symmetric OSp(2|2) WZNW action which will be an action for a "heterotic" string in $\mathrm{AdS}_3 \times \mathrm{S}^1$ space. The parity ($\mathbf{Z}_4$ symmetry), the κ symmetry and the field equations of the action are examined. The κ symmetry variation is quite analogous to the AdS_5 superstring case [2, 13], since the bosonic Sp(2) WZ term does not contribute to the κ transformation. The κ symmetry gauge fixing is necessary to derive chiral right-moving current conervation. This is familiar situation to the Green-Schwarz superstring in flat space where the lightcone gauge is necessary for chiral separation to make the world-sheet superconformal theory. The possible solution of the field equation is proposed. The right-moving mode contains both bosons and fermions, but the left-moving mode contains only bosons. In section 4 the flat limit of our action is examined. The current conservation equations reduce into Klein-Gordon equations representing free right/left-moving bosons. The κ gauge fixing condition and the κ symmetry equation reduce into a Dirac equation representing free right-moving fermions.

2. OSp supergroup

We consider OSp(N|2) as the simplest supergroup containing $\mathrm{SL}(2,R) = \mathrm{Sp}(2)$ which could give a nontrivial WZ term. OSp(N|2) is the 3-dimensional AdS group with N supersymmetry or equally the 2-dimensional N superconformal group. For N=2 its bosonic part is Sp(2) $\times$ SO(2) corresponding to $\mathrm{AdS}_3 \times \mathrm{S}^1$ space and its fermionic part contains four supersymmetries. In this section we present concrete parametrization of Sp(2) and OSp(2|2). Although concrete parametrization is not necessary to examine the κ symmetry and field equations, it is necessary for a concrete expression of the action.

2.1 AdS_3

In general an AdS_d space is described by a coset $\mathrm{SO}(2,d-1)/\,\mathrm{SO}(1,d-1)$. But for $d=3$ case the coset $\mathrm{SO}(2,2)/\,\mathrm{SO}(1,2)$ reduces into $\mathrm{SL}(2) = \mathrm{Sp}(2)$. We choose a Sp(2) group element as

$$X = \frac{1}{\sqrt{1-x^2}}\left(\mathbf{I} + \sum_{m=0,1,2} x^m\gamma_m\right) \quad , \quad X^{-1} = \frac{1}{\sqrt{1-x^2}}\left(\mathbf{I} - \sum_{m=0,1,2} x^m\gamma_m\right) \tag{2.1}$$

where the γ matrix satisfies $\{\gamma_m,\gamma_n\} = 2\eta_{mn}$ with $\eta_{mn} = \mathrm{diag.}(-1,1,\ldots,1)$. It is noted that $\omega_{\alpha\beta}$ is the Sp(2)-invariant metric. So $\gamma_\alpha{}^\beta$ is not symmetric, but $\gamma_\alpha{}^\gamma\omega_{\gamma\beta}$ is symmetric. The LI one form for Sp(2) is given by

$$X^{-1}dX = \frac{1}{1-x^2}\sum_{m=0,1,2}\gamma^m\left(dx_m - \epsilon_{mnl}x^n dx^l\right) \quad . \tag{2.2}$$

The metric for AdS_3 space is obtained as

$$ds^2 = \frac{1}{2}\mathrm{tr}(X^{-1}dX)^2 = \frac{1}{1-x^2}\sum_{m,n=0,1,2} dx^m\left(\eta_{mn} + \frac{x_m x_n}{1-x^2}\right)dx^n \quad . \tag{2.3}$$

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If we generalize to d-dimension, this form of the metric is invariant under the finite SO$(2,d-1) \ni M_{\hat{m}\hat{n}}$ transformation with $\hat{m} = (\natural, m) = (\natural, 0, 1, \dots, d-1)$ and omitting $\natural$ index

$$x_m \to x'_m = \frac{C_m + D_m{}^n x_n}{A + B^l x_l} \quad , \quad M_{\hat{m}}{}^{\hat{n}} = \begin{pmatrix} A & B^n \\ C_m & D_m{}^n \end{pmatrix} \quad . \tag{2.4}$$

A SO$(2, d-1)$ matrix, $M_{\hat{m}}{}^{\hat{n}}$, satisfies

$$(M^T)^{\hat{m}}{}_{\hat{n}} \eta^{\hat{n}\hat{l}} M_{\hat{l}}{}^{\hat{k}} = \eta^{\hat{m}\hat{k}} \quad , \quad \eta^{\hat{m}\hat{n}} = \text{diag.}(-1,-1,1,\dots,1) \tag{2.5}$$

and in components

$$\begin{cases} -A^2 + C_m \eta^{mn} C_n = -1 \\ -B^m B^n + D_l{}^m \eta^{lk} D_k{}^n = \eta^{mn} \\ -AB^m + C_n \eta^{nl} D_l{}^m = 0 \end{cases} \quad . \tag{2.6}$$

The coordinate x_m is a "projective coordinate" of the SO$(2, d-1)$ group realizing the AdS symmetry group by the fractional linear transformation (2.4) as discussed in [19]. Therefore the metric (2.3) has the 2-dimensional conformal group invariance which is SO(2, 2).

2.2 OSp$(N|M)$

In this subsection the general properties of the orthosymplectic supergroup, OSp$(N|M)$, are presented introducing our notation. An OSp$(N|M)$ group element, z, satisfies

$$(z^T)^A{}_B \Omega^{BC} z_C{}^D = \Omega^{AD} \quad , \quad \Omega_{AB} = \Omega^{AB} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \omega \end{pmatrix} \tag{2.7}$$

with $A, B, \ldots = (i, \alpha) = (1, \ldots, N,\ 1, \ldots, M)$. It is denoted that ω is an anti-symmetric metric with $\omega^2 = -\mathbf{1}$, so $\Omega^T \Omega = \mathbf{1}$. The Lie algebra elements $osp(N|M) \ni M_A{}^B$ satisfy

$$\begin{aligned} &(M^T)^A{}_B \Omega^{BD} + \Omega^{AC} M_C{}^D = 0 \\ M_{AB} \equiv M_A{}^C \Omega_{CB} \ \to \ &M_{ij} = -M_{ji}\,,\ M_{\alpha\beta} = M_{\beta\alpha}\,,\ M_{i\alpha} = M_{\alpha i}. \end{aligned} \tag{2.8}$$

The Lie algebra is given by

$$[M_{AB}, M_{CD}\} = \Omega_{[D|[A} M_{B)|C)} \tag{2.9}$$

with a graded commutator; $\mathcal{O}_{[AB)} = \mathcal{O}_{AB} - (-)^{AB} \mathcal{O}_{BA}$ and $[\mathcal{O}, \mathcal{O}'\} = \mathcal{O}\mathcal{O}' - (-)^{\mathcal{O}\mathcal{O}'} \mathcal{O}'\mathcal{O}$. For a group element z the LI one form is given by $L_A{}^B = d\sigma^\mu (L_\mu)_A{}^B = d\sigma^\mu (z^{-1}\partial_\mu z)_A{}^B$ and we use the following notation

$$L_{AB} \equiv L_A{}^C \Omega_{CB} = \begin{pmatrix} \mathbf{L}_{ij} & L_{i\beta} \\ L_{j\alpha} & \mathbf{L}_{\alpha\beta} \end{pmatrix} \quad \text{with } \mathbf{L}_{ij} = -\mathbf{L}_{ji}\,,\ \mathbf{L}_{\alpha\beta} = \mathbf{L}_{\beta\alpha} \quad . \tag{2.10}$$

They satisfy the following Maurer-Cartan equations:

$$\begin{aligned} \epsilon^{\mu\nu} \left[\partial_\mu (\mathbf{L}_\nu)_{ij} + (\mathbf{L}_\mu)_{ik} (\mathbf{L}_\nu)_{kj} - (L_\mu)_{i\alpha} (L_\nu)_{j\beta} \omega^{\alpha\beta} \right] &= 0 \\ \epsilon^{\mu\nu} \left[\partial_\mu (\mathbf{L}_\nu)_{\alpha\beta} + (\mathbf{L}_\mu)_{\alpha\gamma} (\mathbf{L}_\nu)_{\delta\beta} \omega^{\gamma\delta} + (L_\mu)_{i\alpha} (L_\nu)_{i\beta} \right] &= 0 \\ \epsilon^{\mu\nu} \left[\partial_\mu (L_\nu)_{i\alpha} + (\mathbf{L}_\mu)_{ik} (L_\nu)_{k\alpha} - (L_\mu)_{i\beta} (\mathbf{L}_\nu)_{\gamma\alpha} \omega^{\beta\gamma} \right] &= 0 \quad . \end{aligned} \tag{2.11}$$

For the OSp(2|2) group indices run as $i, j = 1, 2$ and $\alpha, \beta = 1, 2$ in the above equations. Denoting $M_{ij} = \epsilon_{ij} T$, $M_{\alpha\beta} = P_{\alpha\beta}$ and $M_{i\alpha} = Q_{i\alpha}$, its Lie algebra $osp(2|2)$ is given by

$$\begin{aligned} &[P_{\alpha\beta}, P_{\gamma\delta}] = \omega_{(\delta|(\alpha} P_{\beta)|\gamma)} \ , && \{Q_{i\alpha}, Q_{j\beta}\} = -\delta_{ij} P_{\alpha\beta} + \omega_{\alpha\beta}\epsilon_{ij} T \\ &[P_{\alpha\beta}, Q_{i\gamma}] = -Q_{i(\alpha}\omega_{\beta)\gamma} \ , && [T, Q_{k\alpha}] = -\epsilon_{kj} Q_{j\alpha} \quad . \end{aligned} \tag{2.12}$$

The Maurer-Cartan equations for $osp(2|2)$ are given by (2.11) without the second term in the first line because $\mathbf{L}_{ik}\mathbf{L}_{kj} \to \mathbf{L}^2 = 0$.

2.3 Left OSp(2|2) invariant one forms

In this section we give a concrete expression of the left OSp(2|2) invariant one forms. We use linear parametrization for the OSp(2|2) matrix instead of familiar exponential parametrization [2]. There is an example of the linear parametrization of OSp supergroup [18], but we use different one as given below.

We parametrize OSp(2|2) group elements as

$$z_A{}^B = \begin{pmatrix} \mathbf{I} & \theta \\ -\omega\theta^T & \mathbf{I} \end{pmatrix} \begin{pmatrix} \Upsilon^{-1/2} & 0 \\ 0 & a^{-1/2}\mathbf{I} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & X \end{pmatrix} \tag{2.13}$$

where $\mathbf{I}$'s are $2{\times}2$ unit matrices. It is convenient to introduce Υ_{ij}'s as

$$\begin{aligned} \Upsilon_{ij} &= \delta_{ij} + \theta_i{}^\alpha \omega_{\alpha\beta} \theta_j{}^\beta \\ \Upsilon^{-1}{}_{ij} &= \delta_{ij} - \frac{1}{a}\theta_i{}^\alpha \omega_{\alpha\beta} \theta_j{}^\beta \\ \Upsilon^{1/2}{}_{ij} &= \delta_{ij} + \frac{1}{1+\sqrt{a}}\theta_i{}^\alpha \omega_{\alpha\beta} \theta_j{}^\beta \\ \Upsilon^{-1/2}{}_{ij} &= \delta_{ij} - \frac{1}{\sqrt{a}(1+\sqrt{a})}\theta_i{}^\alpha \omega_{\alpha\beta} \theta_j{}^\beta \\ a &= 1 - \frac{1}{2}\theta_i{}^\alpha \omega_{\alpha\beta} \theta_i{}^\beta \end{aligned} \tag{2.14}$$

with $\Upsilon^n{}_{ij}\theta_j = a^n \theta_i$. Then the OSp condition, $z^T \Omega z = \Omega$, leads to $Y^T Y = \mathbf{I}$ and $X^T \omega X = \omega$, i.e. $Y \in \mathrm{O}(2)$ and $X \in \mathrm{Sp}(2)$.

The inverse of z is given by

$$z^{-1}{}_A{}^B = \begin{pmatrix} Y^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} \Upsilon^{-1/2} & 0 \\ 0 & a^{-1/2}\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\theta \\ \omega\theta^T & \mathbf{I} \end{pmatrix} \quad . \tag{2.15}$$

The LI one forms, $L_A{}^B = (z^{-1}\partial z)_A{}^B$, are given by

$$\begin{aligned} \mathbf{L}_i{}^j &= \left(Y^{-1}\partial Y\right)_i{}^j \\ &\quad + Y^{-1}{}_i{}^k \left[\frac{1}{4a(1+\sqrt{a})^2} \sum_{m=0,1,2} (\theta\gamma_m\omega\partial\theta)\, \theta_{[k}\gamma^m\omega\theta_{l]} + \frac{1}{\sqrt{a}(1+\sqrt{a})}\theta_{[k}\omega\partial\theta_{l]} \right] Y_l{}^j \\ \mathbf{L}_\alpha{}^\beta &= (X^{-1}\partial X)_\alpha{}^\beta + X^{-1}{}_\alpha{}^\gamma \left[\frac{1}{2a} \sum_{m=0,1,2} (\gamma_m)_\gamma{}^\delta \left(\theta_i \gamma^m \omega \partial\theta_i\right) \right] X_\delta{}^\beta \end{aligned} \tag{2.16}$$

$$L_i{}^\alpha = \left(Y^{-1}\Upsilon^{-1/2}\right)_{ij} \partial\theta_j{}^\gamma \frac{1}{\sqrt{a}} X_\gamma{}^\alpha \tag{2.17}$$

with $\theta_{[k}\omega\gamma^m\theta_{l]} = \theta_{[k}{}^\alpha(\omega\gamma^m)_{\alpha\beta}\theta_{l]}{}^\beta$, $\theta_{[k}\omega\partial\theta_{l]} = \theta_{[k}{}^\alpha\omega_{\alpha\beta}\partial\theta_{l]}{}^\beta$ and $\theta\omega\gamma_m\partial\theta = \theta_k{}^\alpha(\omega\gamma_m)_{\alpha\beta}\partial\theta_k{}^\beta$.

3. κ symmetric OSp(2|2) WZNW action

We consider the following action for a supersymmetric string in the OSp(2|2) background whose bosonic part is $AdS_3 \times S^1$. The criteria to construct an action are followings:

1. it has (pseudo) global OSp(2|2) invariance;
2. its bosonic Sp(2) part is the standard WZNW model;
3. the WZ term is closed, $dH = 0$;
4. it has generalized even parity, or equally $\mathbf{Z}_4$ invariance;
5. it has κ-symmetry invariance;
6. its field equation gives the chiral right-moving current conservation.

We propose the following action:

$$\begin{aligned} S &= S_0 + S_{\rm WZ} \\ S_0 &= \frac{1}{2T}\int d^2\sigma\sqrt{-h}h^{\mu\nu}{\rm Str}\left[(z^{-1}\partial z)\mid_{\text{bosonic part}}\right]^2 \\ &= \frac{1}{2T}\int d^2\sigma\sqrt{-h}h^{\mu\nu}\left[(\mathbf{L}_\mu)_{ji}(\mathbf{L}_\nu)_{ij} - (\mathbf{L}_\mu)_{\alpha\beta}\omega^{\beta\gamma}(\mathbf{L}_\nu)_{\gamma\delta}\omega^{\delta\alpha}\right] \qquad (3.1) \\ S_{\rm WZ} &= \frac{k}{2}\int d^3\sigma H \\ H &= \frac{1}{3}\mathbf{L}_{\alpha\beta}\omega^{\beta\gamma}\mathbf{L}_{\gamma\delta}\omega^{\delta\epsilon}\mathbf{L}_{\epsilon\phi}\omega^{\phi\alpha} - L_i{}^\alpha\mathbf{L}_{\alpha\beta}L_i{}^\beta - L_i{}^\alpha\mathbf{L}_{ij}\omega_{\alpha\beta}L_j{}^\beta \quad . \qquad (3.2) \end{aligned}$$

The criteria 1-3 are guiding principles to determine the above form:

- criterion 1: the OSp(2|2) invariance is manifest up to total derivative caused from the variation of the WZ term as usual, since this action is written in terms of the LI one forms. Furthermore we also impose another global Sp(2) symmetry, $\mathbf{L}_{\alpha\beta} \to (g^T\mathbf{L}g)_{\alpha\beta}$ and $L_{i\alpha} \to (Lg)_{i\alpha}$ for Sp(2) $\in g$. This Sp(2) symmetry corresponds to a part of the AdS_3 isomentry and it is not expressed by "$z \to zg$" type transformation. This Sp(2) together with Sp(2) $\subset$ OSp(2|2) forms SO(2, 2), the AdS_3 or the 2-dimensional conformal group, discussed in the subsection 2.1.

- criterion 2: the bosonic Sp(2) part of the action is obtained by setting $\theta = 0$ and $Y = \mathbf{I}$. The survived X dependence is just standard WZNW model

$$S \to -\frac{1}{2T}\int {\rm tr}(X^{-1}\partial X)^2 + \frac{k}{6}\int {\rm tr}(X^{-1}dX)^3 \quad . \qquad (3.3)$$

- criterion 3: the three form H is determined from the closure, $dH = 0$, using the Maurer-Cartan equations in (2.11) for $osp(2|2)$. It is also mentioned that H in (3.2) can not be rewritten as ${\rm Str}(z^{-1}dz)^3$.

We will show that the action (3.1) and (3.2) satisfies the criteria 4-6 as below.

3.1 $\mathbf{Z}_4$ invariance

A super-AdS group is a "generalized symmetric space" based on the supersymmetrized parity, namely $\mathbf{Z}_4$ symmetry, rather than an usual "symmetric space" [6]. The parity operation is given by $\Pi(M)$ with $\Pi^4(M) = M$. The invariant subalgebra, $\Pi(M) = M$, is $u(1)\times u(1)$ which is denoted by $\mathcal{H}_0$. The $\mathbf{Z}_4$ decomposition of the $osp(2|2)$ algebra is given by

$$\begin{aligned} &\mathcal{H}_0 = \{T_{12},\ P_{12} + P_{21}\}\,, && \mathcal{H}_1 = \{q_1 \pm q_2\} \\ &\mathcal{H}_2 = \{P_{11},\ P_{22}\}\,, && \mathcal{H}_3 = \{q'_1 \pm q'_2\} \end{aligned} \tag{3.4}$$

where we denoted $Q_{i\alpha} = (q_i,\ q'_i)$ with $q_i = Q_{i1}$ and $q'_i = Q_{i2}$. Each subspace satisfies the following algebra $[\mathcal{H}_n, \mathcal{H}_m] \subset \mathcal{H}_{n+m}$ (mod 4).

The $\mathcal{H}_n$ component of the LI currents is denoted by j_n. The action (3.1) and (3.2) is expressed as

$$\begin{aligned} S_0 &\sim \int d^2\sigma\, [j_0 j_0 + j_2 j_2] \\ S_{\mathrm{WZ}} &\sim \int d^3\sigma\, [j_0 \wedge (j_2 \wedge j_2 + j_1 \wedge j_3) + j_2 \wedge (j_1 \wedge j_1 + j_3 \wedge j_3)] \quad . \end{aligned} \tag{3.5}$$

All terms are of even parity or equivalently $\mathbf{Z}_4$ invariant.

Our $\mathbf{Z}_4$ classification (3.4) is not covariant under the other global Sp(2), which is part of 3-dimensional AdS symmetry. In the original classification in [6] $\mathcal{H}_0$ coincides with H for a coset G/H. On the other hand our space is not a coset space, so one might consider an empty $\mathcal{H}_0$. However one of the three $sp(2)$ generators must be $\mathcal{H}_0$ in such a way that the bosonic tri-linear term in the WZ action belongs to $\mathcal{H}_0$; $j_2 \wedge j_2 \wedge j_2 \notin \mathcal{H}_0$. We have chosen $P_{(12)}$ among $sp(2)$ generators, $P_{(\alpha\beta)}$, as $\mathcal{H}_0$.

In general the WZ term $\mathcal{L}_{\mathrm{WZ}}$ has a surface term ambiguity. The OSp(2|2) invariance and the $\mathbf{Z}_4$ invariance restrict ambiguous terms to be a form of dj_0. A candidate term with 3-dimensional AdS symmetry is $d\mathbf{L}_{ij}\epsilon^{ij} = L_i{}^{\alpha} L_j{}^{\beta}\omega_{\beta\alpha}\epsilon^{ij}$. A surface term does not effect the value of three form H, the field equation and the κ gauge variation. The local WZ term, which is a form of fermionic currents bilinears such as $j_1 \wedge j_3$, does not exist for this system except the surface term $L_i{}^{\alpha} L_j{}^{\beta}\omega_{\beta\alpha}\epsilon^{ij} = d\mathbf{L}_{ij}\epsilon^{ij}$. Therefore our WZ term is unique up to this surface term ambiguity.

3.2 κ symmetry invariance

The system has "usual" Virasoro constraints, since the kinetic term of the action in (3.1) contains only bilinears of bosonic LI currents. In general the κ symmetry variation of the action is proportional to the Virasoro constraints so that it is cancelled by the variation of the Virasoro multiplier. When the action is written in terms of the LI currents, the κ symmetry variation is a part of the local right transformation $z \to z\Lambda$ in such a way that the parameter Λ carries the same indices with the LI currents. We will determine the κ symmetry transformation rules by cancellation between the $z \to z\Lambda$ variation of the action and the Virasoro term.

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The LI one form is transformed under $z \to z\Lambda$ as

$$\delta_\Lambda (z^{-1}\partial_\mu z)_A{}^B = \partial_\mu \Lambda_A{}^B + (z^{-1}\partial_\mu z)_A{}^C \Lambda_C{}^B - \Lambda_A{}^C (z^{-1}\partial_\mu z)_C{}^B \quad . \tag{3.6}$$

For a fermionic parameter $\lambda_{i\alpha}$ the LI one form in components are transformed as

$$\begin{aligned} (\delta_\lambda \mathbf{L}_\mu)_{ij} &= -(L_\mu)_{i\alpha}\omega^{\alpha\beta}\lambda_{j\beta} + \lambda_{i\alpha}\omega^{\alpha\beta}(L_\mu)_{j\beta} \\ (\delta_\lambda \mathbf{L}_\mu)_{\alpha\beta} &= (L_\mu)_{i\alpha}\lambda_{i\beta} - \lambda_{i\alpha}(L_\mu)_{i\beta} \\ (\delta_\lambda L_\mu)_{i\alpha} &= \partial_\mu \lambda_{i\alpha} + (\mathbf{L}_\mu)_{ij}\lambda_{j\alpha} + \lambda_{i\beta}\omega^{\beta\gamma}(\mathbf{L}_\mu)_{\gamma\alpha} \quad . \end{aligned} \tag{3.7}$$

It is convenient to introduce $\sqrt{-h}h^{\mu\nu} = \frac{1}{e}e_+{}^{(\mu}e_-{}^{\nu)}$. The variation of the kinetic term is

$$\delta_\lambda \mathcal{L}_0 = \frac{2}{T}\frac{1}{e}e_+{}^{(\mu}e_-{}^{\nu)}\lambda_{i\alpha}\omega^{\alpha\beta}\left((\mathbf{L}_\mu)_{\beta\gamma}\omega^{\gamma\delta}(L_\nu)_{i\delta} - (\mathbf{L}_\mu)_{ij}(L_\nu)_{j\beta}\right) \quad . \tag{3.8}$$

The κ variation of the WZ term is given by

$$\delta_\lambda \mathcal{L}_{\rm WZ} = \frac{k}{e}e_+{}^{[\mu}e_-{}^{\nu]}\lambda_{i\alpha}\omega^{\alpha\beta}\left((\mathbf{L}_\mu)_{\beta\gamma}\omega^{\gamma\delta}(L_\nu)_{i\delta} - (\mathbf{L}_\mu)_{ij}(L_\nu)_{j\beta}\right) \tag{3.9}$$

where $\epsilon^{\mu\nu} = e_+{}^{[\mu}e_-{}^{\nu]}/e$ is used.

We consider the variation $\delta e_\pm{}^\mu = \varphi_{\pm +}e_-{}^\mu + \varphi_{\pm -}e_+{}^\mu$, and so $\delta e = (\varphi_{-+} + \varphi_{+-})e$. Under this variation $(\sqrt{-h}h^{\mu\nu})$ is transformed as

$$\delta_\varphi\left(\sqrt{-h}h^{\mu\nu}\right) = \frac{1}{e}\left(\varphi_{++}e_-{}^{(\mu}e_-{}^{\nu)} + \varphi_{--}e_+{}^{(\mu}e_+{}^{\nu)}\right) \quad , \tag{3.10}$$

and the transformed Virasoro term is given by

$$\delta_\varphi \mathcal{L} = \frac{1}{2T}\frac{1}{e}\left(\varphi_{++}e_-{}^{(\mu}e_-{}^{\nu)} + \varphi_{--}e_+{}^{(\mu}e_+{}^{\nu)}\right)\left((\mathbf{L}_\mu)_{ij}(\mathbf{L}_\nu)_{ji} - (\mathbf{L}_\mu)_\alpha{}^\beta(\mathbf{L}_\nu)_\beta{}^\alpha\right) \tag{3.11}$$

with $(\mathbf{L}_\mu)_\alpha{}^\beta = \omega^{\beta\gamma}(\mathbf{L}_\mu)_{\alpha\gamma}$. In the prefactor $e_-{}^{(\mu}e_-{}^{\nu)}$ and $e_+{}^{(\mu}e_+{}^{\nu)}$ form an orthogonal basis.

The variation of the total Lagrangian is

$$\begin{aligned} \delta\mathcal{L} = \frac{1}{2Te}&\left[\varphi_{++}\left\{(\mathbf{L}_-)_{ij}(\mathbf{L}_-)_{ji} - (\mathbf{L}_-)_\alpha{}^\beta(\mathbf{L}_-)_\beta{}^\alpha\right\} + \varphi_{--}\left\{(\mathbf{L}_+)_{ij}(\mathbf{L}_+)_{ji} - (\mathbf{L}_+)_\alpha{}^\beta(\mathbf{L}_+)_\beta{}^\alpha\right\}\right] \\ &+\left\{\left(\frac{2}{T}+k\right)\mathcal{P}_+^{\mu\nu} + \left(\frac{2}{T}-k\right)\mathcal{P}_-^{\mu\nu}\right\}\lambda_{i\alpha}\omega^{\alpha\beta}\left\{-(\mathbf{L}_\mu)_{ij}(L_\nu)_{j\beta} - (\mathbf{L}_\mu)_\beta{}^\gamma(L_\nu)_{i\gamma}\right\} \end{aligned} \tag{3.12}$$

with the projection operator $\mathcal{P}_\pm^{\mu\nu} \equiv \frac{1}{e}e_\pm{}^\mu e_\mp{}^\nu = \frac{1}{2}(\sqrt{-h}h^{\mu\nu} \pm \epsilon^{\mu\nu})$ and $L_\pm = e_\pm{}^\mu L_\mu$. For a case $k = -\frac{2}{T}$, the $\lambda_{i\alpha}$ parameter is determined from the κ symmetry invariance as

$$\lambda_{i\alpha} = \mathcal{P}_-^{\rho\lambda}\left\{(\mathbf{L}_\rho)_{ij}(\kappa_\lambda)_{j\alpha} - (\mathbf{L}_\rho)_\alpha{}^\beta(\kappa_\lambda)_{i\beta}\right\} = (\mathbf{L}_-)_{ij}(\kappa_+)_{j\alpha} - (\mathbf{L}_-)_\alpha{}^\beta(\kappa_+)_{i\beta} \quad . \tag{3.13}$$

The κ symmetry invariance is obtained as

$$\begin{aligned} \delta\mathcal{L} &= \left(\frac{1}{2Te}\varphi_{++} + \frac{2}{T}(\kappa_+)_{i\alpha}\omega^{\alpha\beta}(L_+)_{i\beta}\right)\left\{(\mathbf{L}_-)_{ij}(\mathbf{L}_-)_{ji} - (\mathbf{L}_-)_\alpha{}^\beta(\mathbf{L}_-)_\beta{}^\alpha\right\} \\ &\quad + \frac{1}{2Te}\varphi_{--}\left\{(\mathbf{L}_+)_{ij}(\mathbf{L}_+)_{ji} - (\mathbf{L}_+)_\alpha{}^\beta(\mathbf{L}_+)_\beta{}^\alpha\right\} = 0 \\ &\Leftrightarrow \varphi_{++} + 4e(\kappa_+)_{i\alpha}\omega^{\alpha\beta}(L_+)_{i\beta} = \varphi_{--} = 0 \quad . \end{aligned} \tag{3.14}$$

If a case $k = \frac{2}{T}$ is chosen instead of $k = -\frac{2}{T}$, then the κ symmetry invariance requires the $\lambda_{i\alpha}$ parameter to be $\lambda_{i\alpha} = (\mathbf{L}_+)_{ij}(\kappa_-)_{j\alpha} - (\mathbf{L}_+)_\alpha{}^\beta(\kappa_-)_{i\beta}$ and $\varphi_{--} + 4e(\kappa_-)_{i\alpha}\omega^{\alpha\beta}(L_-)_{i\beta} = 0 = \varphi_{++}$.

The κ transformation, $\delta z = z \begin{pmatrix} 0 & \lambda \\ -\omega\lambda^T & 0 \end{pmatrix}$, is expressed in components as

$$
\begin{aligned}
\delta\theta_i{}^\alpha &= (\Upsilon^{1/2} Y \lambda X^{-1} a^{1/2})_i{}^\alpha \\
&= (\Upsilon^{1/2} Y)_i{}^j \left((\mathbf{L}_-)_{jk}(\kappa_+)_{k\beta} + (\mathbf{L}_-)_{\beta\gamma}\omega^{\gamma\delta}(\kappa_+)_{j\delta}\right)\left(-\omega X^{-1} a^{1/2}\right)^{\beta\alpha} \\
\left(\delta X X^{-1}\right)_\alpha{}^\beta &= -\frac{1}{2a}\sum_{m=0,1,2}(\gamma_m)_\alpha{}^\beta\left(\theta\gamma^m\omega\delta\theta\right) \qquad (3.15) \\
\left(\delta Y Y^{-1}\right)_{ij} &= -\epsilon_{ij}\left(\frac{1}{\sqrt{a}(1+\sqrt{a})}(\theta_k\omega\epsilon_{kl}\delta\theta_l) + \frac{1}{4a(1+\sqrt{a})^2}(\theta\gamma^m\omega\delta\theta)(\theta_k\gamma_m\omega\epsilon_{kl}\theta_l)\right) \quad ,
\end{aligned}
$$

where spinor indices are omitted; for example $\theta\gamma^m\omega\delta\theta = \theta_i{}^\alpha(\gamma^m\omega)_{\alpha\beta}\delta\theta_i{}^\beta$. If we use the parametrization (2.1) and $Y = e^{i\tau_2 y}$, then the left hand sides of the bosonic equation are given by $(\delta X X^{-1})_\alpha{}^\beta = \sum(\gamma^m)_\alpha{}^\beta\ \delta x^n\left(\eta_{nm} - \epsilon_{nml}x^l\right)/\left(1 - x^2\right)$ and $(\delta Y Y^{-1})_{ij} = \epsilon_{ij}\delta y$.

The Virasoro constraints and the κ symmetry equation are obtained by field equations, $\delta\mathcal{L}/\delta\varphi = 0$ and $\delta\mathcal{L}/\delta\lambda = 0$:

$$
(\mathbf{L}_+)_{ij}(\mathbf{L}_+)_{ji} - (\mathbf{L}_+)_\alpha{}^\beta(\mathbf{L}_+)_\beta{}^\alpha = 0, \qquad (\mathbf{L}_-)_{ij}(\mathbf{L}_-)_{ji} - (\mathbf{L}_-)_\alpha{}^\beta(\mathbf{L}_-)_\beta{}^\alpha = 0 \quad (3.16)
$$

$$
\begin{aligned}
&\left(\frac{2}{T}+k\right)\left\{(\mathbf{L}_+)_{ij}(L_-)_{j\alpha} + (\mathbf{L}_+)_\alpha{}^\beta(L_-)_{i\beta}\right\} \\
&\quad + \left(\frac{2}{T}-k\right)\left\{(\mathbf{L}_-)_{ij}(L_+)_{j\alpha} + (\mathbf{L}_-)_\alpha{}^\beta(L_+)_{i\beta}\right\} = 0
\end{aligned}
$$

Since we have chosen $k = -2/T$, the κ symmetry equation is reduced to

$$
(\mathbf{L}_-)_{ij}(L_+)_{j\alpha} + (\mathbf{L}_-)_\alpha{}^\beta(L_+)_{i\beta} = 0 \quad . \qquad (3.17)
$$

These equations are written in terms of LI currents and they are local equations.

3.3 Chiral current conservations

Now let us compute the chiral current conservations. It was shown that the conserved Noether currents reflecting the global symmetry are RI currents, while the supercovariant derivatives and local constraints are made of LI currents [15, 17]. So we need to consider the infinitesimal variation $\delta z z^{-1}$ which carries the same indices with the RI currents in order to evaluate the current conservations. Under this variation the LI one form is transformed as

$$
\delta\ (z^{-1}\partial_\mu z) = z^{-1}\partial_\mu(\delta z z^{-1})z \quad . \qquad (3.18)
$$

The three form is transformed as

$$
\begin{aligned}
\delta H &= \mathrm{Str}\left[\epsilon^{\rho\mu\nu}\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)(z^{-1}\partial_\mu z)(z^{-1}\partial_\nu z)\right] \\
&= \epsilon^{\rho\mu\nu}\partial_\rho\ \mathrm{Str}\left[\delta z z^{-1}\ \partial_\mu\left\{(\partial_\nu z)z^{-1}\right\}\right] \qquad (3.19)
\end{aligned}
$$

where the explicit expression of the supertrace is given by

$$
\begin{aligned}
&\epsilon^{\rho\mu\nu}\mathrm{Str}\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)(z^{-1}\partial_\mu z)(z^{-1}\partial_\nu z)\\
&\quad=\epsilon^{\rho\mu\nu}\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)_A{}^B(z^{-1}\partial_\mu z)_B{}^C(z^{-1}\partial_\nu z)_C{}^A(-1)^A\\
&\quad=\epsilon^{\rho\mu\nu}\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)_A{}^B(L_\mu)_{BC'}\Omega^{CC'}(L_\nu)_{CA'}\Omega^{AA'}(-1)^A\\
&\quad=\epsilon^{\rho\mu\nu}\Big[\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)_{ij}\left\{-(L_\mu)_{j\alpha}\omega^{\alpha\beta}(L_\nu)_{i\beta}\right\}\\
&\qquad+\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)_\alpha{}^\beta\left\{-(\mathbf{L}_\mu)_{\beta\gamma}\omega^{\gamma\delta}(\mathbf{L}_\nu)_{\delta\epsilon}+(L_\mu)_{i\beta}L_{i\epsilon}\right\}\omega^{\epsilon\alpha}\\
&\qquad+2\left(z^{-1}\partial_\rho(\delta z z^{-1})z\right)_{\alpha j}\left\{(\mathbf{L}_\mu)_{jk}(L_\nu)_{k\delta}-(L_\mu)_{j\delta}\omega^{\beta\gamma}(\mathbf{L}_\nu)_{\gamma\delta}\right\}\omega^{\delta\alpha}\Big].
\end{aligned}
\tag{3.20}
$$

The variation of the kinetic term is given by

$$
\begin{aligned}
\delta\mathcal{L}_0&=\frac{1}{T}\sqrt{-h}h^{\mu\nu}\mathrm{Str}\left[(z^{-1}\partial_\mu z)\mid_{\text{bosonic part}}\left(z^{-1}\partial_\nu(\delta z z^{-1})z\right)\mid_{\text{bosonic part}}\right]\\
&=\frac{1}{T}\sqrt{-h}h^{\mu\nu}\mathrm{Str}\left[\partial_\mu(\delta z z^{-1})\left\{\partial_\nu z z^{-1}-z\begin{pmatrix}0&L_\nu\\L_\nu^T&0\end{pmatrix}\Omega^T z^{-1}\right\}\right]\quad,
\end{aligned}
$$

and the variation of the WZ term is given by

$$
\delta\mathcal{L}_{\mathrm{WZ}}=-\frac{k}{2}\epsilon^{\mu\nu}\mathrm{Str}\partial_\mu(\delta z z^{-1})(\partial_\nu z z^{-1})\quad. \tag{3.21}
$$

Total variation is written as

$$
\begin{aligned}
\delta\mathcal{L}=&\left\{\left(\frac{1}{T}-\frac{k}{2}\right)\mathcal{P}_+^{\mu\nu}+\left(\frac{1}{T}+\frac{k}{2}\right)\mathcal{P}_-^{\mu\nu}\right\}\mathrm{Str}\left[\partial_\mu(\delta z z^{-1})\,(\partial_\nu z z^{-1})\right]\\
&-\frac{1}{T}(\mathcal{P}_+^{\mu\nu}+\mathcal{P}_-^{\mu\nu})\mathrm{Str}\left[\partial_\mu(\delta z z^{-1})\,z\begin{pmatrix}0&L_\nu\\L_\nu^T&0\end{pmatrix}\Omega^T z^{-1}\right]=0\quad.
\end{aligned}
\tag{3.22}
$$

We consider a case $k=-2/T$. If the fermionic one form contribution in the second line is absent, the variation (3.22) reduces into the $\partial_+(\partial_- z z^{-1})=0$ in the conformal gauge, $\mathcal{P}_+^{\mu\nu}=(\eta^{\mu\nu}+\epsilon^{\mu\nu})/2$. The second line contribution is caused from the κ symmetry invariance, and at the same time the κ symmetry constraint ambiguity also exists. In this paper we find the κ gauge fixing in which the chiral current conservation becomes manifest. We take the lightcone gauge $v\neq 0$ for the bosonic LI current $(\mathbf{L}_-)_{\alpha\beta}\omega^{\beta\gamma}=\begin{pmatrix}u&v\\s&t\end{pmatrix}$. Using the κ gauge symmetry in (3.15) as $\delta\theta_{i1}=v(\kappa_+)_{i2}+\cdots$, we take the following gauge for the fermionic current as

$$
(L_+)_{i1}=0\quad. \tag{3.23}
$$

We could take the usual lightcone gauge $\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}\theta_{i1}\\\theta_{i2}\end{pmatrix}=0$ or equivalently $\theta_{i1}=0$ which is similar to the temporal gauge. But we rather choose the gauge condition containing a derivative in (3.23) which may be similar to the Lorentz gauge. In this gauge the equation for the κ symmetry (3.17) is solved as $(L_+)_{i2}=0$. This together with the equation (3.23) reduces into

$$(L_+)_{i\alpha}=0 \quad . \tag{3.24}$$

This equation corresponds to the one for a free right-moving fermion in a flat limit as we will see in the next subsection. Using the condition (3.24) the field equation is obtained from (3.22) as

$$\partial_+(J_-)_A{}^B=0 \;\;,\;\; (J_-)_A{}^B=\partial_-zz^{-1}-\tfrac{1}{2}z\begin{pmatrix}0 & L_-\\ L_-^T & 0\end{pmatrix}\Omega^Tz^{-1}\equiv(\mathcal{D}_-z)z^{-1}\;. \tag{3.25}$$

It seems that the second term of J_- is typical contribution caused from the κ symmetry as seen in the case of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring [14, 15, 17].

We propose a solution of the equations (3.24) and (3.25) as

$$z=Z_{(-)}(x,y,\theta;\;\sigma^-)\tilde{Z}_{(+)}(x,y;\;\sigma^+)\;\;,\;\;\sigma^\pm=\sigma\pm\tau \tag{3.26}$$

where $Z_{(-)}$ is a function of both bosonic and fermionic right-moving coordinates while $\tilde{Z}_{(+)}$ is a function of only bosonic left-moving coordinates such as

$$\tilde{Z}_{(+)}(\sigma^+)=\begin{pmatrix}Y_{(+)}(\sigma^+) & 0\\ 0 & X_{(+)}(\sigma^+)\end{pmatrix}\quad . \tag{3.27}$$

It is straightforward to check the equation for the right-moving currents given in (3.25) as follows. The first term of the right-moving currents (3.25) is

$$\partial_-zz^{-1}=\partial_-Z_{(-)}Z_{(-)}^{-1}\quad . \tag{3.28}$$

The LI one forms are given by

$$\begin{aligned}L_-=z^{-1}\partial_-z\;&=\;\tilde{Z}_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}\tilde{Z}_{(+)}\\ &=\begin{pmatrix}\left(Y_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}Y_{(+)}\right)_i{}^j & \left(Y_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}X_{(+)}\right)_i{}^\beta\\ \left(X_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}Y_{(+)}\right)_\alpha{}^j & \left(X_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}X_{(+)}\right)_\alpha{}^\beta\end{pmatrix}\quad .\end{aligned} \tag{3.29}$$

So the second term of the left-moving currents (3.25) is calculated as

$$\begin{aligned}&z\begin{pmatrix}0 & L_-\\ L_-^T & 0\end{pmatrix}\Omega^Tz^{-1}\\ &=Z_{(-)}\begin{pmatrix}Y_{(+)} & 0\\ 0 & X_{(+)}\end{pmatrix}\begin{pmatrix}0 & Y_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}X_{(+)}\\ X_{(+)}^{-1}Z_{(-)}^{-1}\partial_-Z_{(-)}Y_{(+)} & 0\end{pmatrix}\begin{pmatrix}Y_{(+)}^{-1} & 0\\ 0 & X_{(+)}^{-1}\end{pmatrix}Z_{(-)}^{-1}\\ &=Z_{(-)}\begin{pmatrix}0 & Z_{(-)}^{-1}\partial_-Z_{(-)}\\ Z_{(-)}^{-1}\partial_-Z_{(-)} & 0\end{pmatrix}Z_{(-)}^{-1}\quad .\end{aligned} \tag{3.30}$$

Therefore the right-moving current, satisfying $\partial_+ J_- = 0$, is given as

$$
\begin{aligned}
(J_-)_A{}^B &= (\mathcal{D}_- z) z^{-1} \\
&= \left[\partial_- Z_{(-)} - \frac{1}{2} Z_{(-)} \begin{pmatrix} 0 & Z_{(-)}^{-1} \partial_- Z_{(-)} \\ Z_{(-)}^{-1} \partial_- Z_{(-)} & 0 \end{pmatrix} \right] Z_{(-)}^{-1} \quad .
\end{aligned} \tag{3.31}
$$

The left-moving current, satisfying $\partial_- \tilde{J}_+ = 0$, is given by

$$
\begin{aligned}
(\tilde{J}_+)_A{}^B &= z^{-1} \partial_+ z \\
&= \tilde{Z}_{(+)}^{-1} \partial_+ \tilde{Z}_{(+)} \ = \begin{pmatrix} Y_{(+)}^{-1} \partial_+ Y_{(+)} & 0 \\ 0 & X_{(+)}^{-1} \partial_+ X_{(+)} \end{pmatrix}
\end{aligned} \tag{3.32}
$$

which contains only bosonic components without fermionic coordinate contribution.

4. Flat limit

In the flat limit the $\mathrm{AdS}_3 \times \mathrm{S}^1$ space becomes 3-dimensional Minkowski $\times$ 1-dimensional Euclidean space. It is obtained by the following rescaling

$$
\begin{array}{lll}
x^m \to x^m / R, & y \to y/R, & \theta \to \theta / \sqrt{R} \\
\mathbf{L}_{\alpha\beta} \to \mathbf{L}_{\alpha\beta}/R, & \mathbf{L}_{ij} \to \mathbf{L}_{ij}/R, & L_{i\alpha} \to L_{i\alpha}/\sqrt{R}
\end{array} \tag{4.1}
$$

and taking $R \to \infty$ limit. The LI currents become

$$
\begin{aligned}
(\mathbf{L}_\mu)^m &= \partial_\mu x^m + \frac{1}{2} \theta_i{}^\alpha (\gamma^m \omega)_{\alpha\beta} \partial_\mu \theta_i{}^\beta \\
(\mathbf{L}_\mu)^y &= \partial_\mu y + \frac{1}{2} \theta_i{}^\alpha \omega_{\alpha\beta} \epsilon_{ij} \partial_\mu \theta_j{}^\beta \\
(L_\mu)_i{}^\alpha &= \partial_\mu \theta_i{}^\alpha
\end{aligned} \tag{4.2}
$$

with $(\mathbf{L}_\mu)_{\alpha\beta} = (\gamma_m \omega)_{\alpha\beta} (\mathbf{L}_\mu)^m$ and $(\mathbf{L}_\mu)_{ij} = \epsilon_{ij} (\mathbf{L}_\mu)^y$. The action in (3.1) and (3.2), which is rescaled as $S \to S/R^2$, becomes

$$
\begin{aligned}
S &= S_0 + S_{\mathrm{WZ}} \\
S_0 &= -\frac{1}{T} \int d^2\sigma \sqrt{-h} h^{\mu\nu} \left[(\mathbf{L}_\mu)^m (\mathbf{L}_\nu)_m + (\mathbf{L}_\mu)^y (\mathbf{L}_\nu)^y \right]
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
S_{\mathrm{WZ}} &= \frac{k}{2} \int d^3\sigma \left[-L_i{}^\alpha \mathbf{L}^m (\gamma_m \omega)_{\alpha\beta} L_i{}^\beta - L_i{}^\alpha \mathbf{L}^y \epsilon_{ij} \omega_{\alpha\beta} L_j{}^\beta \right] \\
&= \frac{k}{2} \int d^2\sigma \epsilon^{\mu\nu} \left[\theta_i{}^\alpha (\gamma_m \omega)_{\alpha\beta} \partial_\mu \theta_i{}^\beta \ \partial_\nu x^m + \theta_i{}^\alpha \omega_{\alpha\beta} \epsilon_{ij} \partial_\mu \theta_j{}^\beta \ \partial_\nu y \right] \quad .
\end{aligned} \tag{4.4}
$$

The bosonic tri-linear term disappears from the WZ term. The lack of the four fermi terms in the WZ term is due to the 4-dimensional N=1 chiral spinor property: This system has the following cyclic identity for the spinors, $\chi_i{}^\alpha$, $(\phi^1)_i{}^\alpha$, $(\phi^2)_i{}^\alpha$ and $(\phi^3)_i{}^\alpha$,

$$
\textstyle\sum_{1,2,3 \text{ cyclic}} \left[\left(\chi_i \gamma_m \omega \delta_{ij} \phi^1{}_j \right) \left(\phi^2{}_k \gamma_m \omega \delta_{kl} \phi^3{}_l \right) + \left(\chi_i \omega \epsilon_{ij} \phi^1{}_j \right) \left(\phi^2{}_k \omega \epsilon_{kl} \phi^3{}_l \right) \right] = 0 \tag{4.5}
$$

where the Sp(2) spinor indices are contracted as $\chi_i\gamma_m\omega\delta_{ij}(\phi^1)_j = \chi_i{}^\alpha(\gamma_m\omega)_{\alpha\beta}\delta_{ij}(\phi^1)_j{}^\beta$ for example. After supplying γ-matrices and restructuring the spinor indices, this will be rewritten as a 4-dimensional covariant cyclic identity.

The global supersymmetry of the LI currents (4.2) is in a familiar form

$$\delta\theta_i{}^\alpha = \varepsilon_i{}^\alpha\ ,\ \delta x^m = -\frac{1}{2}\varepsilon_i{}^\alpha(\gamma^m\omega)_{\alpha\beta}\theta_i{}^\beta\ ,\ \delta y = -\frac{1}{2}\varepsilon_i{}^\alpha\omega_{\alpha\beta}\epsilon_{ij}\theta_j{}^\beta \quad . \tag{4.6}$$

The kinetic term (4.3) is manifestly invariant and the WZ term (4.4) is pseudo invariant under this supersymmetry as usual.

The κ symmetry transformations of the action (4.3) and (4.4) for $k = -2/T$ are given by

$$\begin{aligned}
\delta\theta_i{}^\alpha &= -(\mathbf{L}_-)^y\epsilon_{ij}(\kappa_+)_j{}^\alpha - (\mathbf{L}_-)^m(\kappa_+)_i{}^\beta(\gamma_m)_\beta{}^\alpha \\
\delta x^m &= -\frac{1}{2}\theta_i{}^\alpha(\gamma^m\omega)_{\alpha\beta}\delta\theta_i{}^\beta\ , \qquad\qquad \delta y = -\frac{1}{2}\theta_i{}^\alpha\omega_{\alpha\beta}\epsilon_{ij}\delta\theta_j{}^\beta \\
\delta\sqrt{-h}h^{\mu\nu} &= 4\partial_+\theta_i{}^\alpha\omega_{\alpha\beta}(\kappa_+)_i{}^\beta\ e_-{}^{(\mu}e_-{}^{\nu)} \quad .
\end{aligned} \tag{4.7}$$

The Virasoro constraints, field equations for bosons and the κ symmetry equation are given as

$$\begin{aligned}
(\mathbf{L}_+)^m(\mathbf{L}_+)_m + (\mathbf{L}_+)^y(\mathbf{L}_+)^y = 0\ ,\ \ (\mathbf{L}_-)^m(\mathbf{L}_-)_m + (\mathbf{L}_-)^y(\mathbf{L}_-)^y &= 0 \\
\partial_\mu\left(\frac{2}{T}\sqrt{-h}h^{\mu\nu}(\mathbf{L}_\nu)^m + \frac{k}{2}\epsilon^{\mu\nu}\theta_i^\alpha(\gamma^m\omega)_{\alpha\beta}\partial_\nu\theta_i{}^\beta\right) &= 0 \\
\partial_\mu\left(\frac{2}{T}\sqrt{-h}h^{\mu\nu}(\mathbf{L}_\nu)^y + \frac{k}{2}\epsilon^{\mu\nu}\theta_i^\alpha\omega_{\alpha\beta}\epsilon_{ij}\partial_\nu\theta_j{}^\beta\right) &= 0 \\
\left\{\left(\frac{1}{T}+\frac{k}{2}\right)\mathcal{P}_+^{\mu\nu}+\left(\frac{1}{T}-\frac{k}{2}\right)\mathcal{P}_-^{\mu\nu}\right\}\left\{(\mathbf{L}_\mu)^m(\gamma_m\omega)_{\alpha\beta}\partial_\nu\theta_i{}^\beta+(\mathbf{L}_\mu)^y\omega_{\alpha\beta}\epsilon_{ij}\partial_\nu\theta_j{}^\beta\right\} &= 0\ .
\end{aligned} \tag{4.8}$$

For a case $k = -\frac{2}{T}$ the κ symmetry equation in (4.8) is

$$(\mathbf{L}_-)^m(\gamma_m)_\alpha{}^\beta\partial_+\theta_{i\beta} + (\mathbf{L}_-)^y\epsilon_{ij}\partial_+\theta_{j\alpha}\ = 0 \quad . \tag{4.9}$$

Now we take the following gauge: The condition for the bosonic current is $(\mathbf{L}_-)^m(\gamma_m)_1{}^2 = -v \neq 0$. The condition for the fermionic current is $(L_+)_{i1} = \partial_+\theta_{i1} = 0$ using the κ symmetry degree of freedom $\delta\theta_{i1} = v(\kappa_+)_{i2} + \cdots$. Our gauge condition in the flat limit is necessary condition for the conventional lightcone gauge as

$$x_0 + x_1 = v\tau + \text{const.}\ ,\quad \begin{pmatrix}0\ 0\\1\ 0\end{pmatrix}\begin{pmatrix}\theta_{i1}\\\theta_{i2}\end{pmatrix} = 0 \quad . \tag{4.10}$$

The κ equation (4.9) leads to $(L_+)_{i2} = \partial_+\theta_{i2} = 0$. This together with the gauge fixing condition reduces into that the fermion coordinates satisfy the free right-moving Dirac equation,

$$(L_+)_{i\alpha} = \partial_+\theta_{i\alpha} = 0 \quad\rightarrow\quad \theta_{i\alpha}(\sigma^-) \quad . \tag{4.11}$$

The chiral right-moving current in (3.25), satisfying $\partial_+(J_-)_A{}^B = 0$, is given by

$$(J_-)_A{}^B = \begin{pmatrix} \epsilon_{ij}\partial_- y & \frac{1}{2}\partial_-\theta_i{}^\beta \\ -\frac{1}{2}\omega_{\alpha\gamma}\partial_-\theta_j{}^\gamma & (\gamma_m)_\alpha{}^\beta\partial_- x^m \end{pmatrix} \quad . \tag{4.12}$$

The left-moving current $\tilde{J}_+ = z^{-1}\partial_+ z$ is given by

$$(\tilde{J}_+)_A{}^B = \begin{pmatrix} \epsilon_{ij}\partial_+ y & 0 \\ 0 & (\gamma_m)_\alpha{}^\beta\partial_+ x^m \end{pmatrix} \tag{4.13}$$

with the gauge $\partial_+\theta = 0$. It satisfies $\partial_-(\tilde{J}_+)_A{}^B = 0$. The bosonic variables x^m and y satisfy free Klein-Gordon equations for both right/left-moving modes

$$\partial_+\partial_- x^m = 0 = \partial_+\partial_- y \ \rightarrow \ x^m = x^m(\sigma^+) + x^m(\sigma^-) \ , \quad y = y(\sigma^+) + y(\sigma^-) \quad . \tag{4.14}$$

Threfore our model in the flat limit is a 4-dimensional "heterotic" string with the N=1 supersymmetric right-moving sector and the bosonic left-moving sector.

The heterotic Green-Schwarz action in flat space is given by the sum of the usual type I Green-Schwarz action plus the chiral current constraint term as shown in the original paper [20]. On the other hand the chiral current conditions, (3.23) and (3.24), are result of the κ-symmetry gauge in our model. From the fact that our model keeps the chiral structure after the flat limit, the κ-symmetry may be essential for the chiral separation rather than the non-abelian bosonic WZ term. Although the relation between different treatments of the chiral condition is unclear at this stage, our model in the flat space limit corresponds to the 4-dimensional part of the usual critical heterotic string in flat space.

5. Conclusion and discussions

We have proposed a κ symmetric WZNW model for OSp(2|2) supergroup. The kinetic term contains only bosonic current bilinears without fermionic current bilinear. The action contains both the WZ term for the Sp(2) WZNW and the one for the Green-Schwarz superstring. Then we have constructed the chiral non-abelian currents corresponding to the equation (1.2) in the introduction. The non-abelian bosonic Wess-Zumino term does not affect the κ symmetry and κ symmetry transformation rules are similar to the AdS superstring case. It is essential that the κ symmetry and the chiral current conservation are consistent only for the same coefficient of the WZ term. We have chosen the lightcone gauge $(\mathbf{L}_{\mathrm{AdS};-})_1{}^2 \neq 0$ for bosonic coordinates and the κ gauge $(L_+)_{i1} = 0$, then the fermionic field equation gives $(L_+)_{i2} = 0$ in this gauge. The κ gauge condition, $(L_+)_{i1} = 0$, contains derivative operator ∂_+ which corresponds to the Lorentz gauge rather than the temporal gauge. It is a necessary condition for the usual lightcone gauge in the flat limit. This allows us to derive the chiral right-moving currents for all $osp(2|2)$ components. The right-moving current, $(J_-)_A{}^B = (\mathcal{D}_- z)z^{-1}$, satisfying $\partial_+(J_-)_A{}^B = 0$ is derived from the field equation as (3.25). The left-moving current, $(\tilde{J}_+)_A{}^B = z^{-1}\partial_+ z$, satisfying $\partial_-(\tilde{J}_+)_A{}^B = 0$ is obtained as (3.32) from the factorization solution. The factorization is given as $z = Z_{(-)}(\sigma^-)\tilde{Z}_{(+)}(\sigma^+)$, where $Z_{(-)}$ is a function of both bosonic and fermionic right-moving coordinates $x,\ y,\ \theta$

while $\tilde{Z}_{(+)}$ is a function of only bosonic left-moving coordinate $x,\ y$. Therefore this model describes a heterotic string propagating in the $\mathrm{AdS}_3{\times}\mathrm{S}^1$ space.

This system itself is not critical and it is expected to be embedded into some larger system to describe critical string. We have obtained the right-moving current, $(J_-)_A{}^B = (\mathcal{D}_- z)z^{-1}$, which is "right-invariant" (RI) reflecting the global OSp$(2\,|\,2)$ invariance of the action. It was shown that the RI currents satisfy the Poisson bracket for the AdS superalgebra [15] and the stress-energy tensor is given by the supertrace of the square of the RI currents [17] when the AdS superstring action is written in terms of the LI currents. So it may be expected that the right-moving RI current constructs Sugawara form giving the central charge $c_R = 0$ because the dimension of $osp(2|2)$ is zero. The left-moving current has only bosonic parts giving the central charge $c_L = 1 + \frac{3k}{k-2}$. Since the right-moving sector has the κ-symmetry invariance in addition to the reparametrization invariance, the central charge has contributions from the reparametrization ghost and the κ-symmetry ghost whose contribution is unknown so far. If a (chiral) superstring in curved space with the κ symmetry exists, there will exist a critial theory with the κ ghost. Our model may also describe type II string with right/left asymmetry, so that it consistently describes $\mathrm{AdS}_3{\times}\mathrm{S}^1$ part embedded into some larger critical system such as $\mathrm{AdS}_3{\times}\mathrm{T}^4{\times}\mathrm{S}^3$($=\mathrm{AdS}_3{\times}\mathrm{S}^1{\times}\mathrm{T}^3{\times}\mathrm{S}^3$). It is interesting to find more systems into which our model can be embedded, and supergravity solutions corresponding to them.

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